Announcements

1) HW 1 Problems due today

2) HW 2 Up - two parts.
Webwork due Friday
Rest due Tuesday

3) Ford Day - tomorrow
recruitment for Ford
Kochoff Hall
Registration: noon
Presentation: 12:15

Networking: 12:45

Also at

3:00 in Business school,

Michigan Room East

Section 12.5

Lines and Planes

Lines in 2-D

You are used to thinking of a line as the graph of $y = mx + b$.

Suppose $b = 0$.

$y = mx$ is a line through the origin, contains all vectors parallel to $\langle 1, m \rangle$.

We can write these vectors

$$x\langle 1, m \rangle$$

$$= \langle x, xm \rangle$$

If we consider the parametric function

$$f(x) = \langle x, mx \rangle,$$

the graph of the function is just the terminal points of the vectors $\langle x, mx \rangle$ for $x \in \mathbb{R}$. The graph is just the graph of a line $y = mx$!

To capture

$y = mx + b$, add

the vector $\langle 0, b \rangle$

to $\langle x, mx \rangle$!

we get

$\langle x, mx+b \rangle$,

terminal points give

the graph of $y = mx+b$.

Lines in 3-D

You cannot write a line as the graph of a function $z = f(x,y)$!

Use the parametric definition and switch the variable x to t.

Lines through $(0,0,0)$:

Any such line contains a vector $v = \langle x_0, y_0, z_0 \rangle$.

Again, the line is formed by taking terminal points of scalar multiples of v.

Example 1: If ℓ is a line through the origin in $\mathbb{R}^3$ and contains the point $(-3, 0, 5)$, the vector $\langle -3, 0, 5 \rangle$ is on the line.

The graph ℓ is determined by terminal points of your function

$$\boxed{f(t) = t\langle -3, 0, 5 \rangle}$$

If the line goes through the point (x_1, y_1, z_1), just add the vector $\langle x_1, y_1, z_1 \rangle$ to $t\langle x_0, y_0, z_0 \rangle$ to get

$$\boxed{f(t) = t\langle x_0, y_0, z_0 \rangle + \langle x_1, y_1, z_1 \rangle}$$

The graph is all terminal points of the vectors $f(t)$ for $t \in \mathbb{R}$.

If the line determined by the direction of the vector $\langle -3, 0, 5 \rangle$ contains the point $(4, 10, \pi)$, we write the equation as

$$\boxed{f(t) = t\langle -3, 0, 5 \rangle + \langle 4, 10, \pi \rangle}$$

Other Equations

(parametric any symmetric)

The generic equation for a line in 3-D with direction determined by the vector $d = \langle x_0, y_0, z_0 \rangle$ and passing through $(x_1, y_1; z_1)$ is

$$\boxed{f(t) = t\langle x_0, y_0, z_0 \rangle + \langle x_1, y_1, z_1 \rangle}$$

The equation

$$\boxed{f(t) = t\langle x_0, y_0, z_0\rangle + \langle x_1, y_1, z_1\rangle}$$

is called the vector equation for the line ℓ it represents.

We can easily rewrite this in parametric form:

$$\boxed{f(t) = \langle x_1 + t x_0, y_1 + t y_0, z_1 + t z_0\rangle}$$

The symmetric form of the line recovers a formula in x, y, and z - but it needs two equalities!

Write the parametric form and set equal to $\langle x, y, z \rangle$.

$$\langle x, y, z \rangle$$

$$= \langle x_1 + t x_0, y_1 + t y_0, z_1 + t z_0 \rangle$$

Equate coordinates, solve for t!

We get

$$t = \frac{x - x_1}{x_0} \quad (x_0 \neq 0)$$

$$= \frac{y - y_1}{y_0} \quad (y_0 \neq 0)$$

$$= \frac{z - z_1}{z_0} \quad (z_0 \neq 0)$$

If either x_0, y_0, or z_0 is zero, we get (respectively)

$x = x_1$, $y = y_1$, $z = z_1$.

If $x_0, y_0, z_0 \neq 0$,

the symmetric equations

of the line are given by

$$\boxed{\frac{x-x_1}{x_0} = \frac{y-y_1}{y_0} = \frac{z-z_1}{z_0}}$$

Example 2: Consider the line

$$f(t) = t\langle -1, 8, 15\rangle + \langle 2, 1, 6\rangle.$$

Find the parametric and symmetric forms for the line.

parametric:

$$f(t) = \langle 2-t, 1+8t, 6+15t\rangle.$$

Symmetric Equations:

$$\frac{x-2}{-1} = \frac{y-1}{8} = \frac{z-6}{15}, \text{ so}$$

$$\boxed{2-x = \frac{y-1}{8} = \frac{z-6}{15}}$$

Parallel, Intersecting, and Skew Lines

In 2-D, lines either intersect or are parallel.

In 3-D, if two lines don't intersect, it doesn't mean that they are parallel! If the two lines lie on the same plane, then they are either intersecting or parallel.

If two lines on the same plane, and they don't intersect, then the vectors that determine their direction should be parallel.

Example 3: Determine whether the lines

$$f(t) = \langle 2+5t, 3-t, 11+16t \rangle$$

and

$$g(t) = \langle -1+10t, 15-2t, 32t \rangle$$

are parallel or not.

$$f(t) = t\langle 5, -1, 16 \rangle + \langle 2, 3, 11 \rangle$$

$$g(t) = t\langle 10, -2, 32 \rangle + \langle -1, 15, 0 \rangle.$$

Since the direction vectors

$\langle 5, -1, 16 \rangle$ and

$\langle 10, -2, 32 \rangle$ are

parallel, the lines

are parallel.

Observation: if two lines are parallel, we can take their direction vectors to be **identical**:

$$f(t) = \langle 2, 0, 1 \rangle t$$

$$g(t) = \langle 16, 0, 8 \rangle t$$

$$= \langle 2, 0, 1 \rangle 8t$$

Redefine g as

$$g(s) = \langle 2, 0, 1 \rangle s$$

with $8t$.

If two lines are not parallel, then they either intersect or they don't. Two non-parallel, non-intersecting lines will be called skew.

Example 4: Determine whether the lines

$$f(t) = \langle t+5, 2t-6, 8t+3 \rangle$$

and

$$g(t) = \langle 8t-2, t+1, 17t \rangle$$

are intersecting, parallel, or skew.

Direction vector for f: $\langle 1, 2, 8 \rangle$

Direction vector for g: $\langle 8, 1, 17 \rangle$

These vectors are not parallel, so the lines are not parallel. To tell if they are intersecting or skew, replace t with s in the equation for g.

$$g(s) = \langle 8s-2, s+1, 17s \rangle.$$

Set equal to

$$f(t) = \langle t+5, 2t-6, 8t+3 \rangle.$$

x-coordinate

$8s - 2 = t + 5$, so

$t = 8s - 7$

Plug this into y-coordinate

$$s + 1 = 2t - 6$$
$$= 2(8s - 7) - 6$$
$$= 16s - 20.$$

Solving for s,

$15s = 21$, $\boxed{s = \frac{7}{5}}$

If $s = 7/5$, then

$t = 8s - 7$

$= 21/5$

Plug these numbers into the z-coordinates

for f: $8(21/5) + 3$

$= 183/5$

for g: $17(7/5) = \dfrac{119}{5}$

Since $183/5 \neq 119/5$,

the lines are skew.

If these two numbers had been equal, the lines would have intersected.

Planes

Any 3 (non-colinear) points determine a plane!

In 2-D, there is only one plane! In 3-D, there are infinitely many.

Easiest: Coordinate planes

analogs of the coordinate axes in 2-D:

in 2-D, the y-axis is $x=0$, the x-axis is $y=0$.

in 3-D, $x=0$ is an equation that is satisfied by any point $(0, y, z)$.

these points make up a plane!

. .

Similarly, the other coordinate planes are

$y=0$ (all points $(x,0,z)$) and

$z=0$ (all points $(x,y,0)$).

Look at $z=0$.

Rewrite using vectors and the dot product!

$$0 = \langle x, y, z \rangle \cdot \langle 0, 0, 1 \rangle$$

$$= z \checkmark$$

So if $\langle x, y, z \rangle$ is a vector on the plane, it is orthogonal to $\langle 0, 0, 1 \rangle$.

Given a vector

$n = \langle A, B, C \rangle$ that

is nonzero, it determines

a line through the origin

(1-dimensional). There

are 3 dimensions in $\mathbb{R}^3$,

so to make up the rest,

take all vectors $\langle x, y, z \rangle$

orthogonal to $\langle A, B, C \rangle$:

this gives a plane through

the origin!

The equation for the plane is then all points (x,y,z) such that

$$\boxed{\begin{aligned} 0 &= \langle A,B,C\rangle \cdot \langle x,y,z\rangle \\ &= Ax+By+Cz \end{aligned}}$$

If the plane passes through (x_0,y_0,z_0), the equation becomes

$$\boxed{0 = \langle A,B,C\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle}$$

Example 5: Find the equation of the plane orthogonal to $\langle -2, 6, 4\rangle$ passing through $(-8, -1, -e)$

Equation:

$$\boxed{\langle -2, 6, 4\rangle \cdot \langle x+8, y+1, z+e\rangle = 0}$$